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SOME NOTES ON VECTOR ANALYSIS.*

By ARTHUR C. LUNN, The University of Chicago.

1. The Parametric Representation of a Dyadic of Rotation.

If a rigid body undergo any displacement or change of orientation, keeping one point fixed, the rectangular coordinates of any point after the motion, referred to the fixed point as origin, are expressible in terms of the coordinates of the same point before the motion by the equations

$$
\begin{aligned}
x'&=a_{11}x+a_{12}y+a_{13}z,\\
y'&=a_{21}x+a_{22}y+a_{23}z,\\
z'&=a_{31}x+a_{32}y+a_{33}z,
\end{aligned}
\tag{1}
$$

where the coefficients a_{ij} constitute an orthogonal matrix or dyadic, being subject to the conditions

$$
\begin{aligned}
&a_{11}^2+a_{21}^2+a_{31}^2=1, \quad a_{12}a_{13}+a_{22}a_{23}+a_{32}a_{33}=0,\\
&a_{12}^2+a_{22}^2+a_{32}^2=1, \quad a_{13}a_{11}+a_{23}a_{21}+a_{33}a_{31}=0,\\
&a_{13}^2+a_{23}^2+a_{32}^2=1, \quad a_{11}a_{12}+a_{21}a_{22}+a_{31}a_{32}=0,
\end{aligned}
\tag{2}
$$

which are obvious geometrically from the interpretation of the a's, and of these combinations of them, as cosines of angles formed by various pairs of lines; or algebraically from the fact that the relation $x'^2+y'^2+z'^2=x^2+y^2+z^2$ must be an identity in x, y, z. These relations are independent and sufficient, so that three of the coefficients are arbitrary; or, more generally, the nine coefficients may be considered as functions of three arbitrary parameters.

Euler showed that any such displacement could be produced by a single rotation through a certain angle about a determinate axis, and therefore that the coefficients could be expressed in terms of the angle of rotation t

*For the notation and theorems of vector analysis here used, see for example: Gibbs-Wilson, *Vector Analysis*, to which reference is made in following foot-notes.

and the direction-cosines λ, μ, ν of the axis of rotation, these being of course subject to the relation $\lambda^2+\mu^2+\nu^2=1$. The resulting values of the coefficients, arranged as in (1) are

$$
\begin{array}{lll}
\lambda^2+(1-\lambda^2)\cos t, & \lambda\mu(1-\cos t)-\nu\sin t, & \lambda\nu(1-\cos t)+\mu\sin t, \\
\lambda\mu(1-\cos t)+\nu\sin t, & \mu^2+(1-\mu^2)\cos t, & \mu\nu(1-\cos t)-\lambda\sin t, \\
\lambda\nu(1-\cos t)-\mu\sin t, & \mu\nu(1-\cos t)+\lambda\sin t, & \nu^2+(1-\nu^2)\cos t.
\end{array} \tag{3}
$$

These were proved by means of geometric construction by Rodrigues,* who also by the substitution

$$\lambda\tan\frac{t}{2}=\alpha,\quad \mu\tan\frac{t}{2}=\beta,\quad \nu\tan\frac{t}{2}=\lambda, \tag{4}$$

reduced them to the form

$$
\begin{array}{lll}
\dfrac{1+\alpha^2-\beta^2-\gamma^2}{1+\tau^2}, & \dfrac{2\alpha\beta-2\gamma}{1+\tau^2}, & \dfrac{2\alpha\gamma+2\beta}{1+\tau^2}, \\[2ex]
\dfrac{2\alpha\beta+2\gamma}{1+\tau^2}, & \dfrac{1+\beta^2-\gamma^2-\alpha^2}{1+\tau^2}, & \dfrac{2\beta\gamma-2\alpha}{1+\tau^2}, \\[2ex]
\dfrac{2\alpha\gamma-2\beta}{1+\tau^2}, & \dfrac{2\beta\gamma+2\alpha}{1+\tau^2}, & \dfrac{1+\gamma^2-\alpha^2-\beta^2}{1+\tau^2},
\end{array} \tag{5}
$$

(where $\tau=\tan\frac{1}{2}t$, $\tau^2=\alpha^2+\beta^2+\gamma^2$) which shows explicitly their dependence on three arbitrary parameters (α, β, γ), these being geometrically the components of a vector in the direction of the axis of rotation whose length is equal to the tangent of half the angle of rotation.

The following deduction leads to the same expressions by means of integration in series of the vector differential equation which gives the law of distribution of velocity at various points of a rotating rigid body.

Let $\mathbf{u}$ be the constant vector of angular velocity, in the direction of the axis, and $\mathbf{r}$ the vector from the origin to any point, then as a vector the linear velocity at that point is†

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=\mathbf{u}\times\mathbf{r}, \tag{6}$$

where t stands for the time but may conveniently be identified with the angle of rotation by taking the angular velocity as numerically equal to unity, so that $\mathbf{u}$ is the unit-vector whose components are (λ, μ, ν). Then

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*Liouville's *Journal de Mathematique*, Ser. 1, t. V, p. 380.

† *Vector Analysis*. p. 98.

$$\frac{d^2\mathbf{r}}{dt^2}=\mathbf{u}\times\frac{d\mathbf{r}}{dt}=\mathbf{u}\times(\mathbf{u}\times\mathbf{r})=(\mathbf{u}.\mathbf{r})\mathbf{u}-\mathbf{r},$$

(7)

$$\frac{d^3\mathbf{r}}{dt^3}=\left(\mathbf{u}.\frac{d\mathbf{r}}{dt}\right)\mathbf{u}-\frac{d\mathbf{r}}{dt}=-\frac{d\mathbf{r}}{dt},$$

since $\mathbf{u}.\frac{d\mathbf{r}}{dt}=\mathbf{u}.(\mathbf{u}\times\mathbf{r})=0$; and by induction

(8) $$\frac{d^{n+2}\mathbf{r}}{dt^{n+2}}=-\frac{d^n\mathbf{r}}{dt^n}$$

for every $n>0$. Hence Maclaurin's series

$$\mathbf{r}_t=\mathbf{r}_0+\frac{d\mathbf{r}}{dt}.t+\frac{d^2\mathbf{r}}{dt^2}.\frac{t^2}{2\,!}+\ldots$$

reduces to

$$\mathbf{r}_t=\mathbf{r}_0+\Big[(\mathbf{u}.\mathbf{r}_0)\mathbf{u}-\mathbf{r}_0\Big]\left[\frac{t^2}{2\,!}-\frac{t^4}{4\,!}+\ldots\right]$$

$$+\Big[\mathbf{u}\times\mathbf{r}_0\Big]\left[\frac{t}{1\,!}-\frac{t^3}{3\,!}+\ldots\right]$$

where the series admit immediate interpretation, giving

(9) $$\mathbf{r}_t=\mathbf{r}_0+[(\mathbf{u}.\mathbf{r}_0)\mathbf{u}-\mathbf{r}_0][1-\cos t]+[\mathbf{u}\times\mathbf{r}_0]\sin t.$$

Here $\mathbf{r}_0$, $\mathbf{r}_t$ are the vectors from the origin to the same point of the body before and after the rotation, respectively, their components being (x, y, z), (x', y', z'); also $\mathbf{u}.\mathbf{r}_0=\lambda\, x+\mu\, y+\nu\, z$, and the components of $\mathbf{u}\times\mathbf{r}_0$ are $(\mu\, z-\nu\, y,\ \nu\, x-\lambda\, z,\ \lambda\, y-\mu\, x)$. Thus equation (9) is equivalent to the Cartesian equations (1) with coefficients expressed as in (3).

2. A Representation of a Solenoidal Vector.

In the general theory of vector functions of a point in space* it is known that a necessary and sufficient condition that a vector have a vanishing curl is that it admit of representation as a potential vector, or one whose components are the partial derivatives of a single scalar function. Symbolically this means that if $\mathbf{v}=\nabla S$, then curl $\mathbf{v}=0$, and conversely, if curl $\mathbf{v}=0$ then there exists a scalar function S such that $\mathbf{v}=\nabla S$. The present note is concerned with an analogous theorem for a solenoidal vector, or one whose divergence vanishes.

* *Vector Analysis*, Chap. III.

It is known, as readily proved by direct differentiation, that if a vector function be the vector product of two potential vectors it is solenoidal; or symbolically, if $\mathbf{v} = \nabla U \times \nabla V$, then $\operatorname{div} \mathbf{v} = 0$. The following proof establishes the converse theorem, that if $\operatorname{div} \mathbf{v} = 0$ then there exist in general two scalar functions U, V, such that $\mathbf{v} = \nabla U \times \nabla V$.

Let $\mathbf{v}$ be at first any vector function, and u, v two independent integrals of the linear partial differential equation in three independent variables

$$\mathbf{v} . \nabla u = 0, \tag{1}$$

which means geometrically that u, v are two functions such that ∇u and ∇v are not collinear, but are both everywhere perpendicular to $\mathbf{v}$. Hence $\mathbf{v}$ can be written

$$\mathbf{v} = w \nabla u \times \nabla v, \tag{2}$$

where w is some scalar function. Then

$$\operatorname{div} \mathbf{v} = w \operatorname{div}(\nabla u \times \nabla v) + \nabla w . (\nabla u \times \nabla v)$$

where the first term vanishes identically so that

$$\operatorname{div} \mathbf{v} = [\nabla u, \ \nabla v, \ \nabla w], \tag{3}$$

this being analytically the Jacobian or functional determinant of the functions u, v, w, with respect to the variables x, y, z, the Cartesian coordinates.

If now $\mathbf{v}$ be solenoidal, or this Jacobian vanish, a well known theorem shows that w must be a function of u, v; so that

$$\mathbf{v} = f(u, v) \nabla u \times \nabla v. \tag{4}$$

Now let U, V be two functions of u, v; then

$$\nabla U \times \nabla V = J \nabla u \times \nabla v \tag{5}$$

where

$$J = \begin{vmatrix} \dfrac{\partial U}{\partial u} & \dfrac{\partial U}{\partial v} \\ \dfrac{\partial V}{\partial u} & \dfrac{\partial V}{\partial v} \end{vmatrix} . \tag{6}$$

It is clearly possible to choose U, V, which are also integrals of (1), so that J, which is a function of u, v, shall be equal to $f(u, v)$; for instance, by taking $U = u$, $V = \int f(u, v)\, dv$. With any such choice then $\mathbf{v}$ takes the form

(7) $$\mathbf{v} = \nabla U \times \nabla V.$$

A necessary and sufficient condition that a vector function be solenoidal is that it admit of representation as the vector product of two potential vectors.

It should be noted that, when $\mathbf{v}$ is given, U and V are not uniquely determined, so that further conditions may perhaps conveniently be imposed upon them in special cases.

THE POSSIBLE ABSTRACT GROUPS OF THE TEN ORDERS 1909 — 1919.

By DR. G. A. MILLER, University of Illinois.

The real essence of fundamental theorems is frequently exhibited most forcibly by means of illustrative examples, especially when these examples have other elements of interest. The determination of all the possible abstract groups whose orders are equal to the numbers of the ten years 1909—1919 offers numerous instructive illustrations of important theorems, and exhibits some properties of these numbers which are at least of temporary interest.

Since 1909=23.83 is the product of two distinct primes such that the larger diminished by unity is not divisible by the smaller, it results that *the cyclic group of order* 1909 *is the only possible group of this order.* That is, there is only one group whose order is equal to the number of the present year. On the contrary, 1910=2.5.191 is the product of three distinct primes and hence every group of order 1910 contains an invariant subgroup of order 191 and also an invariant subgroup of order 955.* The latter may be either cyclic or non-cyclic, since 191—1 is divisible by 5.

As the group of isomorphisms of the cylic group of order 955 involves three operators of order 2 and the identity, there are four groups of order 1910 which involve a cyclic subgroup of order 955. The group of isomorphisms of the non-cyclic group of order 955 is the holomorph of the group of order 191 and hence it contains only one set of conjugate operators of order 2. When the invariant subgroup of order 955 is non-cyclic an operator of order 2 in the entire group must either transform each operator of this invari nt subgroup into itself or it must transform these operators according to an operator of order 2 in the group of isomorphisms. Hence there are two groups of order 1910 involving a non-cyclic subgroup of the order 955 and *there are exactly six distinct groups of order* 1910; *four of them involve a cyclic subgroup of order* 955 *while the remaining two do not have this property.*

*Cf. Burnside's *Theory of Groups of Finite Order*, 1897, p. 353.